ON THE NUMBER OF MAXIMAL INDEPENDENT SETS IN A GRAPH

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ABSTRACT. Miller and Muller (1960) and independently Moon and Moser (1965) determined the maximum number of maximal independent sets in an n-vertex graph. We give a new and simple proof of this result.

Let G be a (simple, undirected, finite) graph. A set $S \subseteq V(G)$ is *independent* if no edge of G has both its endpoints in S. An independent set S is *maximal* if no independent set of G properly contains S. Let $\mathrm{MIS}(G)$ be the set of all maximal independent sets in G. Miller and Muller [7] and Moon and Moser [8] independently proved that the maximum, taken over all n-vertex graphs G, of $|\,\mathrm{MIS}(G)|$ equals

$$g(n) := \begin{cases} 3^{n/3} & \text{if } n \equiv 0 \pmod 3 \\ 4 \cdot 3^{(n-4)/3} & \text{if } n \equiv 1 \pmod 3 \\ 2 \cdot 3^{(n-2)/3} & \text{if } n \equiv 2 \pmod 3\ . \end{cases}$$

This result is important because $g(n)$ bounds the time complexity of various algorithms that output all maximal independent sets [1–3, 5, 6, 9, 10]. Here we give a new and simple proof of this upper bound on $|\,\mathrm{MIS}(G)|$.

Theorem 1 ([7, 8])**.** *For every n-vertex graph G,*

$$|\,\mathrm{MIS}(G)| \leq g(n)\ .$$

Proof. We proceed by induction on n. The base case with $n \leq 2$ is easily verified. Now assume that $n \geq 3$. Let G be a graph with n vertices. Let d be the minimum degree of G. Let v be a vertex of degree d in G. Let $N[v]$ be the closed neighbourhood of v. If $I \in \mathrm{MIS}(G)$ then $I \cap N[v] \neq \emptyset$, otherwise $I \cup \{v\}$ would

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be an independent set. Moreover, if $w \in I \cap N[v]$ then $I \setminus \{w\} \in \text{MIS}(G - N[w])$. Thus

$$|\,\text{MIS}(G)| \leq \sum_{w \in N_G[v]} |\,\text{MIS}(G - N_G[w])| \ ,$$

Since $\deg(w) \geq d$ and g is non-decreasing, by induction,

$$|\,\text{MIS}(G)| \leq (d+1) \cdot g(n-d-1) \ .$$

Note that

$$4 \cdot 3^{(n-4)/3} \leq g(n) \leq 3^{n/3} \ .$$

If $d \geq 3$ then

$$|\,\text{MIS}(G)| \leq (d+1) \cdot 3^{(n-d-1)/3} \leq 4 \cdot 3^{(n-4)/3} \leq g(n) \ .$$

If $d = 2$ then

$$|\,\text{MIS}(G)| \leq 3 \cdot g(n-3) = g(n) \ .$$

If $d = 1$ and $n \equiv 1 \pmod 3$ then since $n - 2 \equiv 2 \pmod 3$,

$$\text{MIS}(G) \leq 2 \cdot g(n-2) \leq 2 \cdot 2 \cdot 3^{(n-2-2)/3} = 4 \cdot 3^{(n-4)/3} = g(n) \ .$$

If $d = 1$ and $n \equiv 0 \pmod 3$ then since $n - 2 \equiv 1 \pmod 3$,

$$\text{MIS}(G) \leq 2 \cdot g(n-2) \leq 2 \cdot 4 \cdot 3^{(n-2-4)/3} < 3^{n/3} = g(n) \ .$$

If $d = 1$ and $n \equiv 2 \pmod 3$ then since $n - 2 \equiv 0 \pmod 3$,

$$\text{MIS}(G) \leq 2 \cdot g(n-2) \leq 2 \cdot 3^{(n-2)/3} = g(n) \ .$$

This proves that $|\,\text{MIS}(G)| \leq g(n)$, as desired. □

For completeness we describe the example by Miller and Muller [7] and Moon and Moser [8] that proves that Theorem 1 is best possible. If $n \equiv 0 \pmod 3$ then let M_n be the disjoint union of $\frac{n}{3}$ copies of K_3. If $n \equiv 1 \pmod 3$ then let M_n be the disjoint union of K_4 and $\frac{n-4}{3}$ copies of K_3. If $n \equiv 2 \pmod 3$ then let M_n be the disjoint union of K_2 and $\frac{n-2}{3}$ copies of K_3. Observe that $|\,\text{MIS}(M_n)| = g(n)$.

Note that Vatter [11] independently proved Theorem 1, and gave a connection between this result and the question, "What is the largest integer that is the product of positive integers with sum n?" Also note that Dieter Kratsch proved that $|\,\text{MIS}(G)| \leq 3^{n/3}$ using a similar proof to that presented here; see Gaspers [4, page 177]. Thanks to the authors of [4, 11] for pointing out these references.

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